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Oscillation Criteria for a Linear Riemann–Stieltjes Integral Equation System*

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1. INTRODUCTION

The system treated here is a type of linear vector Riemann–Stieltjes integral equation system, which under certain conditions reduces to a classical second-order system.

An associated functional is introduced in Section 2, and the relationship between disconjugacy of the system and this functional is explored. In Section 3, further relationships between the functional and disconjugacy are treated. The relationship between focal points and a similar functional is discussed in Section 4. The Morse Quadratic Form for this system is introduced in Section 5 and the relationship between this form and focal points is discussed therein; the relationship between this form and conjugate points is explored in Section 6.

We shall be concerned with the system

$$\begin{aligned} u(t) &= u_0 + \int_a^t [dN]v, \\ v(t) &= v_0 + \int_a^t [dM]u, \end{aligned} \qquad \text{for} \quad t \in [a, b], \tag{E}$$

where M and N are $n \times n$ dimensional complex valued matrix functions, while u and v are n-dimensional complex valued vector functions. The author has previously studied this system in [1] and whenever any results

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are used from that paper they will be denoted by the addition of the Roman numeral I (for example, Theorem I.1.1). The terminology of that paper will be used here. In particular, *we will assume M and N satisfy* H *and* H_h of that paper, which are given here for ready reference.

H. *M and N are of bounded variation and N is continuous on* $[a, b]$.

H_h. *$M(t)$ and $N(t)$ are hermitian for* $t \in [a, b]$.

2. An Associated Functional

We take the following sets:

$$L[a, b] = \{\zeta \mid \zeta \text{ is an } n\text{-dimensional vector function which is integrable with respect to } N\};$$

$$D[a, b] = \{\eta \mid \text{there exists a function } \zeta \in L[a, b] \text{ such that } L_2[\eta, \zeta] = 0\},$$

where $[a, b]$ is a compact interval and

$$L_2[\eta, \zeta] = d\eta - [dN]\zeta. \tag{2.1}$$

The relationship between η and ζ will be indicated by $\eta \in D[a, b] : \zeta$.

If $(\eta_\alpha ; \zeta_\alpha) \in D[a, b] \times L[a, b]$ $(\alpha = 1, 2)$, let $J[\eta_1 : \zeta_1 , \eta_2 : \zeta_2 ; a, b]$ denote the functional defined by

$$J[\eta_1 : \zeta_1 , \eta_2 : \zeta_2 ; a, b] = \int_a^b \zeta_2^*[dN]\, \zeta_1 + \int_a^b \eta_2^*[dM]\, \eta_1 . \tag{2.2}$$

If M and N satisfy H_h, then (2.2) defines an hermitian form on $D[a, b] \times L[a, b]$; that is, if $(\eta_\alpha ; \zeta_\alpha) \in D[a, b] \times L[a, b]$, $(\alpha = 1, 2, 3)$, then

(a) $J[\eta_1 : \zeta_1 , \eta_2 : \zeta_2 ; a, b] = \overline{J[\eta_2 : \zeta_2 , \eta_1 : \zeta_1 ; a, b]}$,

(b) $J[c\eta_1 : c\zeta_1 , \eta_2 : \zeta_2 ; a, b] = cJ[\eta_1 : \zeta_1 , \eta_2 : \zeta_2 ; a, b]$,

(c) $J[\eta_1 + \eta_2 : \zeta_1 + \zeta_2 , \eta_3 : \zeta_3 ; a, b] = J[\eta_1 : \eta_3 , \zeta_1 : \zeta_3 ; a, b] + J[\eta_2 : \eta_3 , \zeta_2 : \zeta_3 ; a, b]$.

In general, for a given η the corresponding vector function ζ is not unique. However, the value of (2.2) is independent of the choice ζ satisfying $\eta \in D[a, b] : \zeta$; for this reason, we shall write (2.2) as

$$J[\eta_1 , \eta_2 ; a, b] = \int_a^b \zeta_2^*[dN]\, \zeta_1 + \int_a^b \eta_2^*[dM]\, \eta_1 . \tag{2.3}$$

Also, for brevity we write $J[\eta; a, b]$ for $J[\eta, \eta; a, b]$.

If we let

$$L_1[\eta, \zeta] = -d\zeta + [dM]\eta,$$

the following result is a ready consequence of the above definitions.

LEMMA 1.1. *If* $\eta_\alpha \in D[a, b] : \zeta_\alpha$, $(\alpha = 1, 2)$, *then*

$$J[\eta_1, \eta_2; a, b] = \eta_2{}^*\zeta_1 |_a^b + \int_a^b \eta_2{}^* L_1[\eta_1, \zeta_1]; \tag{2.4'}$$

$$J[\eta_1; a, b] = \eta_1{}^*\zeta_1 |_a^b + \int_a^b \eta_1{}^* L_1[\eta_1, \zeta_1]; \tag{2.4''}$$

$$\begin{aligned} \int_a^b \eta_2{}^* L_1[\eta_1, \zeta_1] - \int_a^b (L_1[\eta_2, \zeta_2])^* \eta_1 &= \{\zeta_2{}^*\eta_1 - \eta_2{}^*\zeta_1\}|_a^b \\ &= \{\eta_1; \zeta_1 \mid \eta_2; \zeta_2\}|_a^b . \end{aligned} \tag{2.4'''}$$

From this we see that if $t_1, t_2 \in [a, b]$ are conjugate and (u, v) is a solution of (E) with $u(t_1) = 0 = u(t_2)$ and $u \not\equiv 0$ on $[t_1, t_2]$, then $(\eta(t), \zeta(t))$ is defined by $(u(t); v(t))$ on $[t_1, t_2]$ and identically zero elsewhere, are functions such that $\eta \in D_0[a, b] : \zeta$ and (2.4″) implies

$$J[\eta; a, b] = J[u; t_1, t_2] = 0.$$

COROLLARY 2.1. *There are no points* $t_1, t_2 \in [a, b]$ *which are conjugate if the only* $\eta \in D_0[a, b]$ *such that* $J[\eta; a, b] = 0$ *is* $\eta(t) \equiv 0$.

THEOREM 2.1. *If* u *is continuous and of bounded variation on* $[a, b]$, *then there exists a* v *such that* $(u; v)$ *is a solution of* (E) *on* $[a, b]$ *if and only if there exists a* $v_1 \in L[a, b]$ *such that* $u \in D[a, b] : v_1$ *and*

$$J[u : v_1, \eta : \zeta; a, b] = 0 \qquad \text{for all } \eta \in D_0[a, b]. \tag{2.5}$$

If $(u; v)$ is a solution of (E) on $[a, b]$ and $\eta \in D_0[a, b]$, then $u \in D[a, b] : v$ and (2.5) is a consequence of (2.4′) for $(\eta_1; \zeta_1) = (u; v)$, $(\eta_2; \zeta_2) = (\eta; \zeta)$.

On the other hand suppose $u \in D[a, b] : v_1$ and (2.5) holds. If $v_0(t)$ is defined by

$$v_0(t) = \int_a^t [dM]u, \tag{2.6}$$

then substituting into (2.5) and using integration by parts we obtain

$$\int_a^b \zeta^*[dN][v_1 - v_0] = 0 \qquad \text{if} \quad \zeta \in L[a, b] \quad \text{and} \quad \int_a^b \zeta^*[dN] = 0. \tag{2.7}$$

By a well-known result of functional analysis (see, for example, [8, p. 138]), if we restrict ζ to be a continuous function, we have that there exists a constant vector λ such that

$$\int_a^b \zeta^*[dN][v_1 - v_0] = \int_a^b \zeta^*[dN]\lambda, \qquad \text{for } \zeta \text{ continuous.}$$

Consequently, $\int_a^t [dN][v_1 - v_0 - \lambda] \equiv 0$ on $[a, b]$. If $v(t) = v_0(t) + \lambda$, we have that $\int_a^t dv = \int_a^t [dM]u$, $t \in [a, b]$ and $u \in D[a, b] : v$, so that $(u; v)$ is a solution of (E).

COROLLARY 2.2. *If $J[\eta; a, b]$ is nonnegative definite on $D_0[a, b]$, and u is an element of $D_0[a, b]$ satisfying $J[u; a, b] = 0$, then there exists a $v \in L[a, b]$ such that $(u; v)$ is a solution of* (E) *on $[a, b]$. In particular, if $u(t) \not\equiv 0$, a and b are conjugate.*

If $\eta \in D_0[a, b]$, we have that $u + \sigma\eta \in D_0[a, b]$ for arbitrary σ so that

$$\begin{aligned} 0 &\leqslant J[u + \sigma\eta; a, b], \\ &= J[u; a, b] + \sigma J[u, \eta; a, b] + \sigma J[\eta, u; a, b] + |\sigma|^2 J[\eta; a, b]. \end{aligned}$$

As $J[u; a, b] = 0$, we can make the right-hand side negative unless $J[u, \eta; a, b] = 0$.

COROLLARY 2.3. *If $J[\eta; a, b]$ is nonnegative definite on $D_0[a, b]$ and $(u; v)$ is a solution of* (E), *while $u_0 \in D[a, b]$ with $u_0(a) = u(a)$, $u_0(b) = u(b)$, then $J[u_0; a, b] \geqslant J[u; a, b]$; moreover, if $J[\eta; a, b]$ is positive definite on $D_0[a, b]$ the inequality holds with equality only if $u_0(t) \equiv u(t)$.*

Preliminary to the necessary and sufficient conditions for the system (E) to be disconjugate on a subinterval of $[a, b]$, the following result will be stated without proof, as it may be established by direct substitution.

LEMMA 2.2. *Suppose that $U(t)$, $V(t)$ are $n \times r$ matrix functions of bounded variation on $[a, b]$ with U continuous. If η_α is continuous and of bounded variation, $\zeta_\alpha \in L[a, b]$, for $\alpha = 1, 2$, and there exists an r-dimensional vector function $h_\alpha(t)$, such that h_α is of bounded variation and continuous on $[a, b]$ while $\eta_\alpha(t) \equiv U(t)\, h_\alpha(t)$, then on this interval we have the identity*

$$\begin{aligned} \int_a^t \{\zeta_2^*[dN]\,\zeta_1 + \eta_2^*[dM]\,\eta_1\} = \int_a^t \{&[\zeta_2 - Vh_2]^*\,[dN][\zeta_1 - Vh_1] \\ &- h_2^*V^*L_2[\eta_1, \zeta_1] - [L_2[\eta_2, \zeta_2])^*\,Vh_1 \\ &+ h_2^*(V^*L_2[U, V] + U^*L_1[U, V]\,h_1 \\ &- h_2^*[U^*V - V^*U][dh_1] + d[h_2^*U^*Vh_1]. \end{aligned}$$

COROLLARY 2.4. *If the column vectors of $Y(t) = (U(t); V(t))$ form an r-dimensional conjoined family of solutions of* (E), *while $\eta \in D[a, b] : \zeta$ and there exists a function $h(t)$ which is continuous and of bounded variation such that $\eta(t) = U(t)\,h(t)$ for $t \in [a, b]$, then*

$$J[\eta; a, b] = \eta^*Vh\,|_a^b + \int_a^b [\zeta - Vh]^*\,[dN][\zeta - Vh].$$

THEOREM 2.2. *If $J[\eta; a, b]$ is nonnegative definite on $D_0[a, b]$, then $N(t)$ is a nondecreasing function.*

If $N(t)$ is not nondecreasing, then there exists an interval $[c, d]$, and a vector ξ, with $|\xi| = 1$, and such that $\xi^*[N(d) - N(c)]\xi = \int_c^d \xi^*[dN]\xi < 0$. Also, there exists a $k_1 > 0$ such that

$$\int_c^d \xi^*[dN]\xi = -k_1\nu[c, d], \tag{2.8}$$

where $\nu[c, d]$ is the variation of N on $[c, d]$. For any $\delta > 0$ there must exist an interval $[e, f] \subset [c, d]$ with $|f - e| < \delta$, and such that

$$\int_e^f \xi^*[dN]\xi \leqslant -k_1\nu[e, f].$$

In particular, if m is a positive integer such that $2^m > n$, we find an interval $[e, f]$ such that (2.8) holds, and

$$\nu[e, f] < k_1/(2^m V[M]) \tag{2.9}$$

where $V[M]$ is the variation of M on $[a, b]$.

Partition $[e, f]$ into $e = t_0 < t_1 < \cdots < t_{2m} = f$ such that

$$\int_{t_{i-1}}^{t_i} \xi^*[dN]\xi = -k_1\nu[e, f]/2^m. \tag{2.10}$$

Then, if $\eta(t)$ is defined as

$$\eta(t) = \int_e^t [dN]\, \xi\phi, \qquad \phi(t) = \Sigma_i c_i \chi_i(t),$$

with χ_i the characteristic function of $[t_{i-1}, t_i]$, and the c_i chosen so that $\sum_i |c_i| = 1$ and $\eta(d) = 0$, then $J[\eta; a, b] < 0$, which is a contradiction to the nonnegative definiteness of J.

In the same manner as that of Reid [7, pp. 326–328], the following result can be established.

THEOREM 2.3. *Suppose $J[\eta; a, b]$ is positive definite on $D_0[a, b]$. If $d[a, b] = d$, Δ is a basis for $\Lambda[a, b]$ with $\Delta^*\Delta = E_d$, and R is an $n \times (n - d)$ matrix such that $R^*\Delta = 0$ and $[\Delta \;\; R]$ is nonsingular, then there exists a unique solution $Y_b(t) = (U_b(t); V_b(t))$ of (E_{n-d}) such that*

$$U_b(a) = R, \qquad U_b(b) = 0, \qquad V_b{}^*(a)\Delta = 0. \tag{2.11}$$

The column vectors of $Y_b(t)$ *form a basis for a conjoined family of solutions of* (E) *of dimension* $n - d$, *and if* $Y_4(t) = (U_4(t); V_4(t))$ *is a second solution of* (E_{n-d}) *whose column vectors form a basis for a conjoined family of solutions of* (E) *of dimension* $n - d$ *and satisfying*

$$U_4(a) = R, \qquad V_4{}^*(a)\Delta = 0, \qquad U_4{}^*(a)\, V_4(a) > U_b{}^*(a)\, V_b(a), \quad (2.12)$$

then $U_4(t)$ *is of rank* $n - d$ *on* $[a, b]$. *Moreover, if* $Y_2(t) = (U_2(t); V_2(t))$ *is the solution of* (E_d) *satisfying the initial conditions* $U_2(a) = \Delta$, $V_2(a) = 0$, *then*

$$Y(t) = ([U_2(t)\ U_4(t)]; [V_2(t)\ V_4(t)]) = (U(t); V(t))$$

is a conjoined basis for (E) *with* $U(t)$ *nonsingular on* $[a, b]$.

THEOREM 2.4. *The form* $J[\eta; a, b]$ *is positive definite on* $D_0[a, b]$ *if and only if* $N(t)$ *is a nondecreasing matrix function on* $[a, b]$ *and there exists a conjoined basis* $Y(t) = (U(t); V(t))$ *for (E) with* U *nonsingular on* $[a, b]$.

If $J[\eta; a, b]$ is positive definite on $D_0[a, b]$, then theorems 2.2 and 2.3 imply that $N(t)$ is nondecreasing on $[a, b]$ and the existence of a conjoined basis $Y(t) = (U(t); V(t))$ for (E) with $U(t)$ nonsingular on $[a, b]$. Conversely, if such a basis exists, then in view of Lemma 2.2 we have for $\eta \in D_0[a, b] : \zeta$ that

$$J[\eta; a, b] = \int_a^b [\zeta - Vh]^* [dN][\zeta - Vh],$$

with $h(t) = U^{-1}(t)\, \eta(t)$. But N is a nondecreasing hermitian matrix function, which implies that

$$K[\alpha; a, b] = \int_a^b \alpha^* \, dN\alpha$$

is a nonnegative definite hermitian form on the vector space of functions which are N-integrable. Thus, if

$$\int_a^b [\zeta - Vh]^* [dN][\zeta - Vh] = 0,$$

we must have

$$\int_a^t [dN][\zeta - Vh] \equiv 0 \qquad \text{for} \quad t \in [a, b].$$

As $L_2[\eta; \zeta] = 0$ and $L_2[U, V] = 0$, it follows that

$$\int_a^t U[dh] = \int_a^t [dN][\zeta - Vh] \equiv 0.$$

Also, since $\int_a^t U\,dh \equiv 0$ implies $\int_a^t dh \equiv 0$, and the condition $\eta(a) = 0$ implies that $h(a) = 0$, it follows that $h(t) \equiv 0$, and $\eta(t) \equiv 0$. Consequently, $J[\eta; a, b]$ is positive definite on $D_0[a, b]$.

THEOREM 2.5. *The form $J[\eta; a, b]$ is positive definite on $D_0[a, b]$ if and only if $N(t)$ is nondecreasing on $[a, b]$ and there is no point $t_1 \in (a, b]$ conjugate to a.*

If $N(t)$ is nondecreasing and a has no conjugate point on $(a, b]$ and $c = \sup\{t \in [a, b] : J[\eta; a, t]$ is positive definite on $D_0[a, t]\}$, then the result follows if $c = b$. If $c < b$, one may show $J[\eta; a, c]$ is nonnegative definite on $D_0[a, c]$ by taking limits of the forms $J[\eta; a, c - \delta]$. Then Corollary 2.2 gives that $J[\eta; a, c]$ is positive definite on $D_0[a, c]$. Thus Theorem 2.4 gives J is positive definite on $D_0[a, c + \epsilon]$ and we have a contradiction to our choice of c.

If the roles of $t = a$ and $t = b$ are interchanged, one may establish the following result.

COROLLARY 2.5. *The form $J[\eta; a, b]$ is positive definite on $D_0[a, b]$ if and only if $N(t)$ is nondecreasing on $[a, b]$, and there is no value on $[a, b)$ which is conjugate to $t = b$.*

3. DISCONJUGACY CRITERIA

The results of the preceding section will be compressed here for ready reference.

THEOREM 3.1. *If $N(t)$ is nondecreasing for $t \in [a, b]$, then the following conditions are equivalent.*

(i) (E) *is disconjugate on $[a, b]$.*

(ii) *$J[\eta; a, b]$ is positive definite on $D_0[a, b]$.*

(iii) *There is no point on $(a, b]$ conjugate to $t = a$.*

(iv) *There is no point on $[a, b)$ conjugate to $t = b$.*

(v) *There exists a conjoined basis $Y(t) = (U(t); V(t))$ for* (E) *with $U(t)$ nonsingular on $[a, b]$.*

(vi) *There exists an $n \times n$ hermitian matrix function $W(t)$, $t \in [a, b]$, which is a solution of the Riccati integral equation*

$$R[W](t) \equiv \int_a^t [dW] + \int_a^t W[dN]W - \int_a^t dM = 0, \qquad t \in [a, b]. \quad (3.1)$$

Suppose that for $\alpha = 1, 2$ the matrix functions M_α and N_α satisfy hypotheses H and $\mathrm{H_h}$. The corresponding classes $D[a, b]$ and $D_0[a, b]$ will be denoted by $D_\alpha[a, b]$ and $D_{\alpha 0}[a, b]$. If we have

$$N_1(t) \equiv N_2(t), \tag{3.2}$$

then $D_1[a, b] = D_2[a, b]$, and $D_{10}[a, b] = D_{20}[a, b]$. However, these relations may occur without (3.2) holding. For $\alpha = 1, 2$ we have the corresponding systems

$$\begin{aligned} L_1{}^\alpha[u, v](t) &= -dv(t) + [dM_\alpha(t)]\, u(t) = 0, \\ L_2{}^\alpha[u, v](t) &= du(t) - [dN_\alpha(t)]\, v(t) = 0, \end{aligned} \tag{3.3_α}$$

and corresponding functionals

$$J_\alpha[\eta, \zeta; a, b] = \int_a^b \{\zeta^*[dN_\alpha]\zeta + \eta^*[dM_\alpha]\eta\}. \tag{3.4_α}$$

In particular, if $D_1[a, b] = D_2[a, b] = D[a, b]$ then the difference functional

$$J_{12}[\eta; a, b] = J_1[\eta; a, b] - J_2[\eta; a, b] \tag{3.5}$$

is well defined for $\eta \in D[a, b]$.

THEOREM 3.2. *Suppose that for* $\alpha = 1, 2$, *the* $n \times n$ *matrix functions* $N_\alpha(t)$, $M_\alpha(t)$ *satisfy hypotheses* H *and* $\mathrm{H_h}$, *and* $N_2(t)$ *is nondecreasing. Also, suppose* $D_1[a, b] = D_2[a, b]$ *and* $J_{12}[\eta; a, b]$ *is nonnegative definite on* $D_0[a, b] = D_{10}[a, b] = D_{20}[a, b]$. *If* (3.3_2) *is disconjugate on* $[a, b]$, *then* (3.3_1) *is also disconjugate on* $[a, b]$. *Moreover, if* $J_{12}[\eta; a, b]$ *is positive definite on* $D_0[a, b]$ *then the solutions of* (3.3_2) *oscillate more rapidly than the solutions of* (3.3_1) *in the following sense: If* t_1 *and* t_2 *are mutually conjugate with respect to* (3.3_1), *then any conjoined basis* $Y(t) = (U(t); V(t))$ *for* (3.3_2) *is singular at least once on* (t_1, t_2).

If (3.3_2) is disconjugate on $[a, b]$, then (ii) of Theorem 3.1 implies that $J_2[\eta; a, b]$ is positive definite on $D_0[a, b]$ so that $J_1[\eta; a, b]$ is positive definite on $D_0[a, b]$. Thus (i) of Theorem 3.1 implies that (3.3_1) is disconjugate on $[a, b]$.

Now, it can be shown that if there is a conjoined basis $Y(t) = (U(t); V(t))$ of (3.3_2) with $U(t)$ nonsingular on (a, b) and $N_2(t)$ is nondecreasing on this interval, then $J_2[\eta; a, b]$ is nonnegative definite on $D_0[a, b]$. Let $u(t)$ be a solution of (3.3_1) with $u(t_1) = 0 = u(t_2)$, and $u(t) \not\equiv 0$ on $[t_1, t_2]$, where $a \leqslant t_1 < t_2 \leqslant b$. If $\eta(t) = u(t)$ for $t \in [t_1, t_2]$, $\eta(t) \equiv 0$ on $[a, t_1] \cup [t_2, b]$, then $\eta \in D_0[a, b]$ and $J_1[\eta; a, b] = J_1[u; t_1, t_2] = 0$, so that $J_2[\eta; a, b] < 0$. Hence, any conjoined basis $Y(t) = (U(t); V(t))$ of (3.3_2) must have at least one point on (t_1, t_2) where $U(t)$ is singular.

THEOREM 3.3. *If $N(t)$ is nondecreasing on $[a, b]$, then* (E) *is disconjugate on $[a, b]$ if and only if one of the following conditions holds.*

(i) *There exists on $[a, b]$ a nonsingular $n \times n$ matrix function $U \in D[a, b] : V$ with V of bounded variation on $[a, b]$ while $\{U; V \mid U; V\}(t) \equiv 0$, and $\int_a^t U^* L_1[U, V]$ is nondecreasing for $t \in [a, b]$.*

(ii) *There exists an $n \times n$ hermitian matrix function $W(t)$ of bounded variation on $[a, b]$ which is such that*

$$R[W](t) = \int_a^t [dW] + \int_a^t W[dN]W - \int_a^t [dM]$$

is nonincreasing for $t \in [a, b]$.

If (E) is disconjugate on $[a, b]$ then there is a conjoined basis $Y(t) = (U(t); V(t))$ of (E) with $U(t)$ nonsingular on $[a, b]$; also, $U(t)$ satisfies (i) and $W(t) = V(t)\, U^{-1}(t)$ satisfies (ii).

On the other hand, if $U(t)$ satisfies (i) then let $P(t) = \int_a^t U^* L_1[U, V]$. Since $U(t)$ is continuous, the integral exists and defines a matrix function of bounded variation on $[a, b]$. If we take the system (3.3_2) to be such that

$$dN_2(t) \equiv dN(t) \qquad dM_2(t) \equiv dM(t) - U^{*-1}(t)[dP(t)]\, U^{-1}(t),$$

then $(U; V)$ is a conjoined basis for (3.3_2). If (3.3_1) is system (E), then

$$J_{12}[\eta; a, b] = \int_a^b \eta^* U^{*-1}[dP]\, U^{-1}\eta \geqslant 0$$

for $\eta \in D_0[a, b]$, so that Theorem 3.2 implies that (E) is disconjugate on $[a, b]$.

Under the condition (ii) if $\Psi(t) = R[W](t)$, then $\Psi(t) \in BV[a, b]$ and Ψ is nonincreasing. If we take $U(t)$ to be the solution of the system

$$\int_a^t dU = \int_a^t [dN(s)]\, W(s)\, U(s), \qquad U(a) = E,$$

and $V(t) = W(t)\, U(t)$, then U and V are $n \times n$ matrix functions on $[a, b]$ with V of bounded variation on $[a, b]$; moreover, $U \in D[a, b] : V$, U is nonsingular on $[a, b]$, while $\{U; V \mid U; V\}(t) \equiv 0$, $t \in [a, b]$, and

$$\int_a^t U^* L_1[U, V] = - \int_a^t U^*[d\Psi]U,$$

which is nondecreasing on $[a, b]$, so we have reduced case (ii) to case (i).

Results may be obtained corresponding to the results of Reid [7, pp. 341–344] concerning sufficient conditions for the existence of principal solutions and properties of solutions when a principal solution exists.

4. Focal Points

We shall denote by $D_{*0}[a, b]$ the class of all $\eta \in D[a, b]$ with $\eta(b) = 0$ and by $D_{0*}[a, b]$ the class of all $\eta \in D[a, b]$ with $\eta(a) = 0$. Then $D_0[a, b] = D_{*0}[a, b] \cap D_{0*}[a, b]$. We shall also consider the functional

$$\hat{J}[\eta_1 : \zeta_1 , \eta_2 : \zeta_2 ; a, b] = \eta_2^*(a) \Gamma \eta_1(a) + J[\eta_1 , \eta_2 ; a, b]. \tag{4.1}$$

If M and N satisfy $\mathrm{H_h}$ and Γ is a hermitian matrix, then $J[\eta_1 : \zeta_1 , \eta_2 : \zeta_2 ; a, b]$ is a hermitian form on $D[a, b] \times L[a, b]$. As in the case of the functional $J[\eta; a, b]$, if $\eta_\alpha \in D[a, b] : \zeta_\alpha$, $(\alpha = 1, 2)$, then the value of (4.1) is independent of the value of ζ_α so that we will abbreviate to $\hat{J}[\eta_1 , \eta_2 ; a, b]$ or $\hat{J}[\eta_1 ; a, b]$ if $\eta_1 = \eta_2$.

Using the results of Theorem 2.1 and Corollary 2.2 we can obtain the following results.

Theorem 4.1. *There exists a solution $(u; v)$ of* (E) *such that*

$$\Gamma u(a) - v(a) = 0, \tag{4.2}$$

if and only if there exists a $v_1 \in L[a, b]$ such that $u \in D[a, b] : v_1$ and $\hat{J}[u; v_1 , \eta : \zeta; a, b] = 0$ for $\eta \in D_{*0}[a, b] : \zeta$.

Corollary 4.1. *If $\hat{J}[\eta; a, b]$ is nonnegative definite on $D_{*0}[a, b]$, and there exists a $u \in D_{*0}[a, b]$ satisfying $\hat{J}[u; a, b] = 0$, then there exists a v such that (u, v) is a solution of* (E) *on $[a, b]$ which satisfies the condition*

$$\Gamma u(a) - v(a) = 0. \qquad u(b) = 0. \tag{4.3}$$

Since Γ is hermitian, the solution $Y(t) = (U(t); V(t))$ which satisfies $Y(a) = (E, \Gamma)$ is a conjoined basis. The following result is proved in a manner similar to that used to establish Theorem 2.5.

Theorem 4.2. *The functional $\hat{J}[\eta; a, b]$ is positive definite on $D_{*0}[a, b]$ if and only if $N(t)$ is nondecreasing on $[a, b]$, and the conjoined basis $Y(t) = (U(t); V(t))$ for* (E) *satisfying $Y(a) = (E; \Gamma)$ is such that $U(t)$ is nonsingular on $[a, b]$.*

Relative to the functional (4.1), or relative to system (E) with initial condition (4.2), a value $\tau \in [a, b]$ is a *right-hand* {*left-hand*} *focal point to $t = a$* if $\tau > a$ $\{\tau < a\}$ and there is a solution $(u(t); v(t))$ of (E) which satisfies (4.2), has $u(\tau) = 0$, and $u(t) \not\equiv 0$ on the interval with a and τ as endpoints.

The following result can be established by an argument similar to that occurring in the proof of Theorem 3.2, and using the result of Theorem 4.2.

THEOREM 4.3. *Suppose that for* $\alpha = 1, 2$ *the* $n \times n$ *matrix functions* $M_\alpha(t)$ *and* $N_\alpha(t)$ *satisfy hypotheses* H *and* $\mathrm{H_h}$, *while* $N_2(t)$ *is nondecreasing on* $[a, b]$. *Moreover, for arbitrary* $[c, d] \subset [a, b]$ *we have* $D_1[c, d] = D_2[c, d] = D[c, d]$, *and* $\Gamma_\alpha(\alpha = 1, 2)$ *are hermitian matrices such that*

$$\begin{aligned}\hat{J}_{12}[\eta; a, b] &= \hat{J}_1[\eta; a, b] - \hat{J}_2[\eta; a, b] \\ &= \eta^*(a)[\Gamma_1 - \Gamma_2]\,\eta(a) + J_{12}[\eta; a, b]\end{aligned}$$

is nonnegative definite on $D_{*0}[a, b]$. *If relative to* $\hat{J}_2[\eta; a, b]$ *there is no right-hand focal point to a on* $(a, b]$, *then relative to* $\hat{J}_1[\eta; a, b]$ *there is also no right-hand focal point to a on* $(a, b]$.

5. THE MORSE QUADRATIC FORM

The results of this section correspond to the results found in [7, pp. 356–366] and the proofs of the results are in most cases the direct analog of Reid's proofs. We wish to examine the relationship of the Morse Quadratic Form and the idea of focal points as defined in Section 4. That is, if system (E) is identically normal (that is, the only solution of the form $(0, v(t))$ is $v(t) \equiv 0$) and $Y(t) = (U(t); V(t))$ is a conjoined basis for (E) on $[a, b]$, then c is a *focal point of the family of order* k if $U(c)$ is singular and of rank $n - k$. The following lemma is basic to the study of these points.

LEMMA 1.1. *Suppose* (E) *is identically normal, and* (E) *is disconjugate on* $[a, b]$. *If* $Y(t) = (U(t); V(t))$ *is a conjoined basis for* (E), *then on* $(a, b]$ *and* $[a, b)$ *there are at most n focal points, each point being counted a number of times equal to its order. Moreover, the focal points of a conjoined basis are isolated.*

Throughout the remainder of this paper we will assume that (E) *is identically normal.*

A partition

$$a = t_0 < t_1 < \cdots < t_m < t_{m+1} = b \tag{5.1}$$

will be called a *fundamental partition* if (E) is disconjugate on each of the subintervals $[t_{i-1}, t_i]$, $i = 1,\ldots, m + 1$. Such a partition exists since, in view of the results of Corollary I.2.1 and Theorem 3.1, there exists a $\delta > 0$ such that if $|c - d| < \delta$, $[c, d] \subset [a, b]$, then (E) is disconjugate on $[c, d]$. Moreover, if $T = \{t_0, t_1,\ldots, t_m, t_{m+1}\}$ is a fundamental partition, then any refinement is also a fundamental partition.

If T is a fundamental partition, then in view of the condition of identical normality and the result of Lemma I.6.1 we have a unique solution $u = u_{\xi j}$, $v = v_{\xi j}$ of (E) such that $u_{\xi j}(t_{j-1}) = \xi_{j-1}$, $u_{\xi j}(t_j) = \xi_j$ $(j = 1, 2,\ldots, m + 1)$, where the ξ_j are aribtrary n-dimensional vectors. If ξ is defined to be the $n(m+2)$ vector

$$\xi = (\xi^{(\rho)}), \qquad \rho = 1, 2, \ldots, n(m+2),$$

with $\xi^{(nj+\alpha)} = \xi_{\alpha j}$ ($\alpha = 1, \ldots, n$, $j = 0, \ldots, m+1$), then the corresponding vector function

$$u_\xi(t) = u_{\xi j}(t), \qquad t_{j-1} \leqslant t \leqslant t_j \ (j = 1, \ldots, m+1),$$

is continuous on $[a, b]$ and linear in the components of ξ. We shall denote by $S(\Pi)$, the set of all vectors ξ. If $\xi_{m+1} = 0$ we shall say $\xi \in S_{*0}(\Pi)$, and if $\xi_0 = 0$ we shall say $\xi \in S_{0*}(\Pi)$. Moreover, set $S_0(\Pi) = S_{0*}(\Pi) \cap S_{*0}(\Pi)$. If G is an $n \times n$ hermitian matrix, the form

$$Q_*^0[\xi^1, \xi^2 \mid \Pi] = \xi_0^{2*} G \xi_0^1 + J[u_{\xi 1}, u_{\xi 2}\,; a, b] \tag{5.2}$$

is hermitian on $S_{*0}[\Pi]$ since J is a hermitian functional. Thus, there is an $n(m+1)$ dimensional, hermitian matrix Q_*^0 such that

$$Q_*^0[\xi^1, \xi^2 \mid \Pi] = \xi^{2*} Q_*^0 \xi^1.$$

THEOREM 5.1. *If G is an $n \times n$ hermitian matrix and T and $u_\xi(t)$ are specified as above, then Q_*^0 is of rank $n(m+1) - r$ if and only if $t = b$ is a focal point of order r of the conjoined family of solutions $Y(t) = (U(t); V(t))$ of* (E) *with $Y(a) = (E; G)$. Moreover, the elements of Q_*^0 are continuous functions of the elements of G and of $t_1, t_2, \ldots, t_m$.*

For the systems of ordinary differential equations considered in [7; Chap. VII, Sect. 7] the proof of a result corresponding to that of the above theorem uses the continuity of the vector functions u_ξ and v_ξ as functions of $t, t_0, t_1, \ldots, t_{m+1}$. In the present situation the functions u_ξ are continuous functions, but the functions v_ξ are not necessarily continuous. However, the type of argument used by Reid [6, pp. 716–717] to establish the stated result for a system (E) where $N(t)$ is absolutely continuous is still valid for the more general problem considered here.

The dimension of the null space

$$\{\xi \mid Q_*^0 \xi = 0\}$$

is called the *nullity of* Q_*^0, and the dimension of the largest subspace on which Q_*^0 is negative definite is called the (*negative*) *index of* Q_*^0. We can now obtain the following results.

THEOREM 5.2. *If Π is a fundamental partition of $[a, b]$, then the index of $Q_*{}^0[\xi \mid \Pi]$ is equal to the number of points on the open interval (a, b) which are right-hand focal points to $t = a$ relative to the functional $\hat{J}[\eta; a, b]$ where each focal point is counted a number of times equal to its order.*

THEOREM 5.3. *If Π is a fundamental partition of $[a, b]$, then the index {index plus nullity} of $Q_*{}^0[\xi \mid \Pi]$ is equal to the largest nonnegative integer k such that there exists a k-dimensional manifold in $D_{*0}[a, b]$ on which $\hat{J}[\eta; a, b]$ is negative definite {nonpositive definite}.*

For a conjoined basis $Y_0(t) = (U_0(t); V_0(t))$ of (E), the designation of a point c where $U_0(c)$ is singular as a focal point is consistent with the characterization of a focal point in terms of the functional J. If $t = a$ is a point such that $U_0(a)$ is nonsingular then $W_0(a) = V_0(a)\, U_0^{-1}(a)$ is hermitian, and $(U(t); V(t)) = (U_0(t)\, U_0^{-1}(a); V_0(t)\, U_0^{-1}(a))$ is a conjoined solution which satisfies $U(a) = E$, $V(a) = V_0(a)\, U_0^{-1}(a)$. If we let $\Gamma = V_0(a)$, then a value $c > a$ will be a focal point of $\hat{J}[\eta; a, b]$ of order k if and only if $U(c)$ is singular of order $n - k$.

For a given $c \in [a, b]$, the points of $[a, b]$ which are right-hand focal points to $t = c$ relative to the functional $\hat{J}[\eta; a, b]$ will be ordered as a sequence $\tau_p{}^+(\Gamma)$, $(p = 1, 2,...)$, and numbered so that $\tau_p{}^+(\Gamma) \leqslant \tau_{p+1}^+(\Gamma)$, with each repeated a number of times equal to its order as a focal point. For focal points we have the following basic separation theorem.

THEOREM 5.4. *Suppose that* (E) *is identically, normal, and for* $\alpha = 1, 2$, *let*

$$\hat{J}_\alpha[\eta; a, b] = \eta^*(a)\, \Gamma_\alpha \eta(a) + \int_a^b \{\zeta^*[dN]\zeta + \eta^*[dM]\eta\},$$

where Γ_1 and Γ_2 are $n \times n$ hermitian matrices. Moreover, let P and N denote the number of positive and negative proper values of the hermitian matrix $\Gamma_1 - \Gamma_2$, where each proper value is repeated a number of times equal to its multiplicity. If for a positive integer p the focal point $\tau_{p+P}^+(\Gamma_2)$ exists, then $\tau_p{}^+(\Gamma_1)$ exists and $\tau_p{}^+(\Gamma_1) \leqslant \tau_{p+P}^+(\Gamma_2)$; if $\tau_{p+N}^+(\Gamma_1)$ exists then $\tau_p{}^+(\Gamma_2)$ exists and $\tau_p{}^+(\Gamma_2) \leqslant \tau_{p+N}^+(\Gamma_1)$.

6. CONJUGATE POINTS

If we take fundamental partitions as in the last section, and $\xi_i \in S_0(\Pi)$, $(i = 1, 2)$, then we again obtain a form $Q^0[\xi^1, \xi^2 \mid \Pi]$ which is fundamental to the study of conjugate points. Using the same techniques as in Section 5, we may establish results corresponding to Theorems 5.1–5.4, along with the following additional results.

THEOREM 6.1. *The number of points on (a, b), $\{(a, b]\}$, conjugate to a is the same as the number of points on (a, b), $\{[a, b)\}$ conjugate to b, where each point is counted a number of times equal to its order as a conjugate point.*

If we let $t_p^+(a)$ and $t_p^-(a)$ be the pth right and left conjugate point of a, respectively, again with the usual order and numbering convention, we get the following results.

THEOREM 6.2. *If $t_p^+(c)$, $\{t_p^-(c)\}$, exists for $c = c_0$, then there exists a $\delta > 0$ such that $t_p^+(c)$, $\{t_p^-(c)\}$ exists for $c \in (c_0 - \delta, c_0 + \delta)$; moreover, $t_p^+(c)$, $\{t_p^-(c)\}$ is continuous at c_0.*

THEOREM 6.3. *If $a_\alpha \in [a, b]$ $(\alpha = 1, 2)$, and $a_1 < a_2$, then whenever $t_p^+(a_2)$, $\{t_p^-(a_1)\}$ exists, the conjugate point $t_p^+(a_1)$, $\{t_p^-(a_2)\}$ also exists and $t_p^+(a_2) > t_p^+(a_1)$, $\{t_p^-(a_2) > t_p^-(a_1)\}$.*

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